HIGHER ORDER DERIVATIVES

OR CUBE, OR ALGEBRAIC, OR INTEGRAL

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kodeketa eta kriptografiaren egunak
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Outline

The Algebraic Normal Form

$f : \mathbb{F}_2^n \to \mathbb{F}_2$ Then f can be uniquely represented as an element of

$$\mathbb{F}_2[x_0, \ldots, x_{n-1}]/(x_0^2 - x_0, \ldots, x_{n-1}^2 - x_{n-1})$$

That is a sum of monomials, i.e. for some $u \in \mathbb{F}_2^n$

$$x^u = \prod_{i=0}^{n-1} x_i^{u_i}$$

Example : $x_0 x_2 x_3 = x^{10110000}$

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$$f(x_0, \ldots, x_{n-1}) = \bigoplus_{u \in \mathbb{F}_2^n} c_u x^u$$

with $c_u \in \mathbb{F}_2$.

Truth table and Monomials

$$f(x_0,\ldots,x_{n-1}) = \bigoplus_{u\in\mathbb{F}_2^n} c_u x^u$$

$$f(a) = \bigoplus_{u\prec a} c_u \text{ and } c_u = \bigoplus_{a\prec u} f(a)$$

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Where $a \prec u$ iff $\mathrm{supp}(a) \subset \mathrm{supp}(u)$ Also

$$\mathrm{wt}(u) = \#\mathrm{supp}(u)$$

FUNCTIONS

A function from $\mathbb{F}_2^n$ to $\mathbb{F}_2^m$ is represented as a collection of m boolean functions, called component functions.

- For permutations, the monomial $x_0 x_1 \cdots x_{n-1}$ never appears
- A random function has its monomials appearing each with probability $1/2$ in each component function.

REPRESENTATION

DEGREE

DIVISION PROPERTY

ATTACK STRATEGIES

RANDOM DIRECTIONS

Higher-order differential attacks [Lai 94]

$$f(x_0, \ldots, x_{n-1}) = \bigoplus_{u \in \mathbb{F}_2^n} c_u x^u$$

Definition (Multivariate degree)

$$d = \deg(f) = \max\{\mathrm{wt}(u), c_u = 1\}$$

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Distinguisher :

For all linear space V, with $\dim(V) \geq d + 1$,

$$g : x \mapsto \sum_{v \in V} f(x + v)$$

is constant to zero.

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At different level :

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- For any F and G, $\deg(F \circ G) \leq \deg(F) \times \deg(G)$
- A better bound by A. Canteau and C. Boura [2011, FSE]
- Better upper bound when the structure is specific [CGGLRS, 2020]

Going further

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A component of $E_k(x)$ can be represented as

$$\sum_{u \in \mathbb{F}_2^n} c_u(k) x^u$$

and assume that for any $v \succ u$, $c_v(k) = 0$.

Then what is the possible degree of $E_k(x)$ in x?

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Upper bound is not enough : lower bound [HLLT 2020]

Specific Higher order derivative

Assume that for any $w \succ u$, $c_w(k) = 0$, then let $V = \{v, v \prec u\}$. Then for any x,

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Division Property [Todo 2015]

TECHNIQUES

- Using the representation of the Sbox and the linear layer, this division property can be used for iterated construction
- Mixed Integer Linear Programming
- Lower bound the degree
- Monomial prediction, monomial trails

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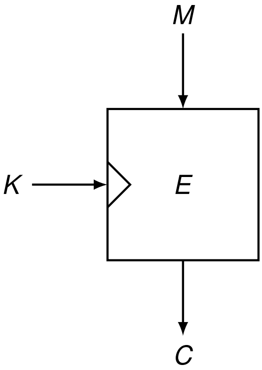
- Not linearly equivalent [LDF, 2020]

BLOCK CIPHERS



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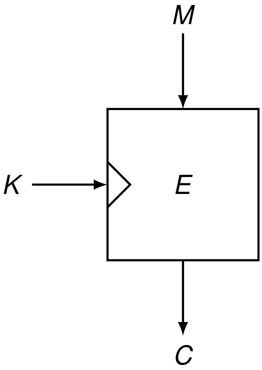

- Proofs of modes, wrt indistinguishability
- Same reasoning for permutation-based constructions.

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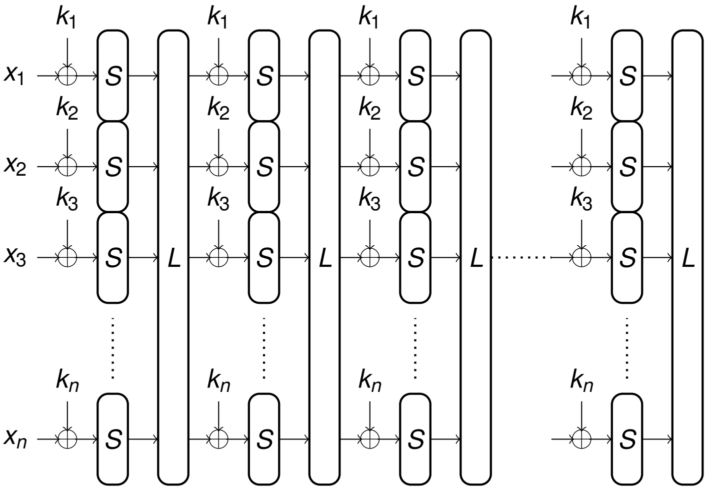

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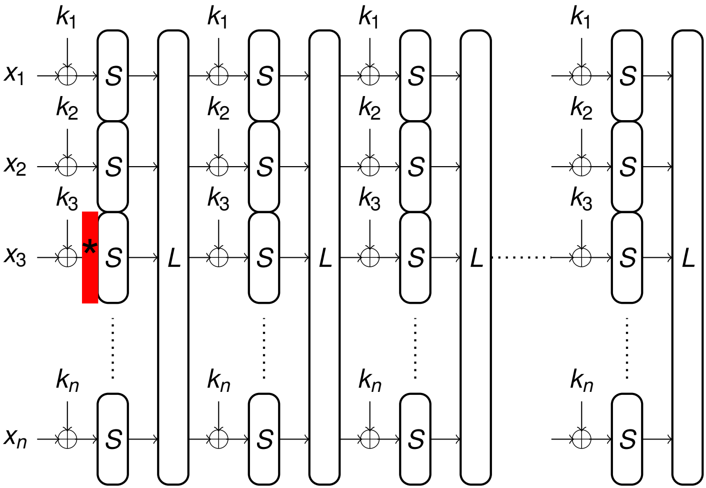

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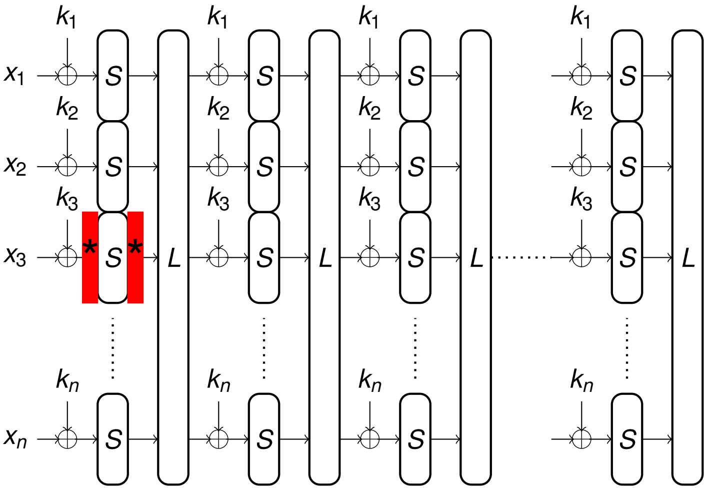

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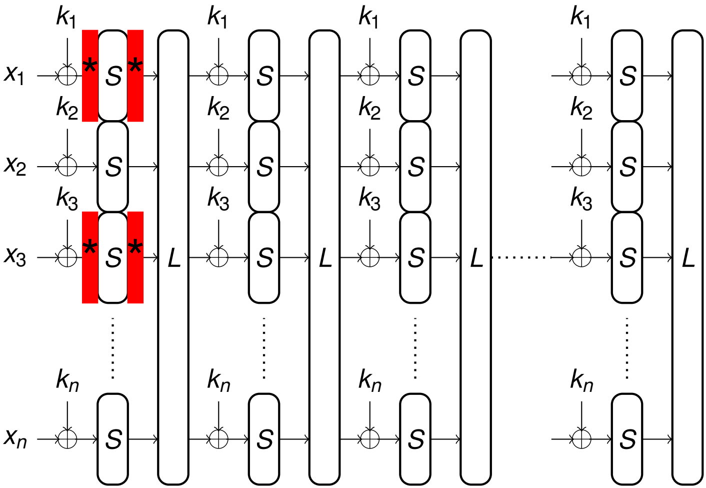

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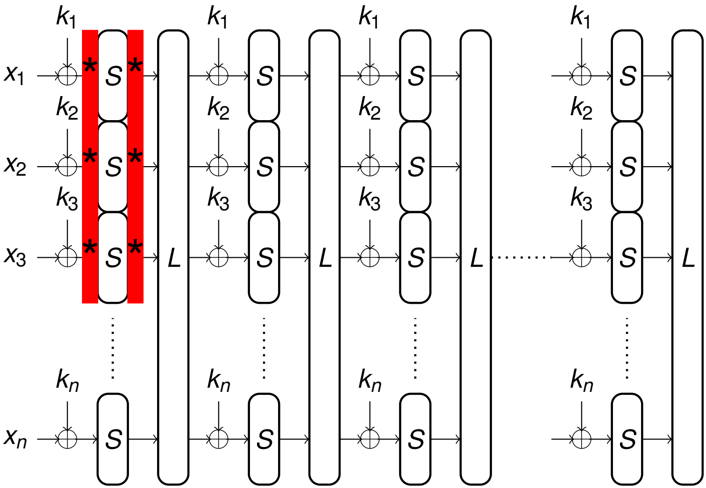

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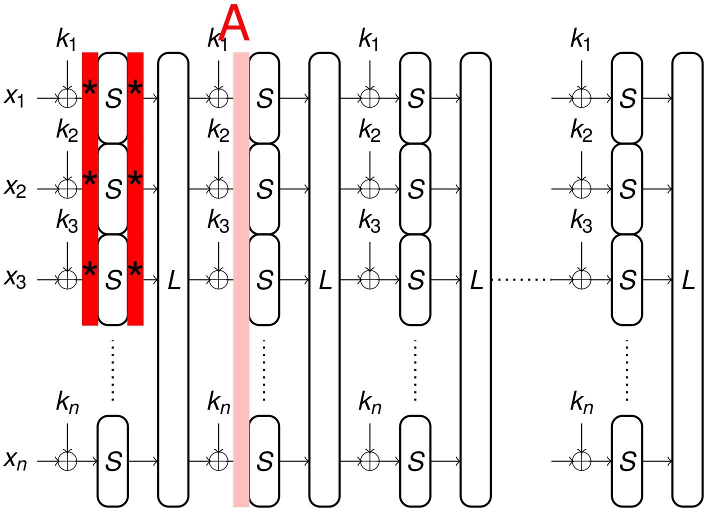

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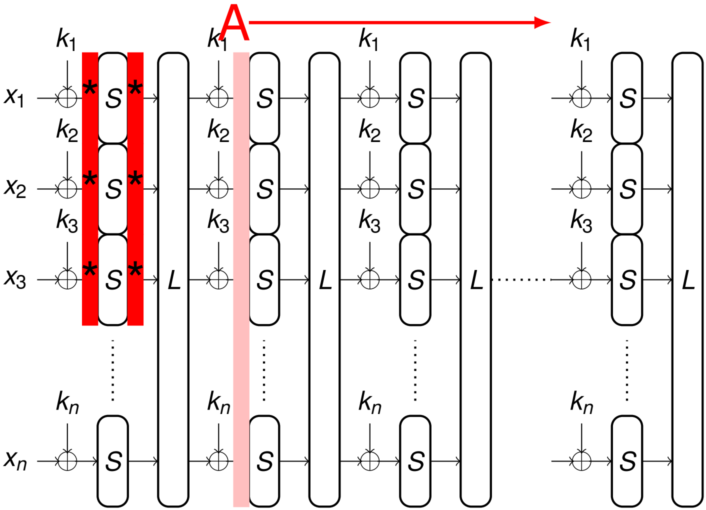

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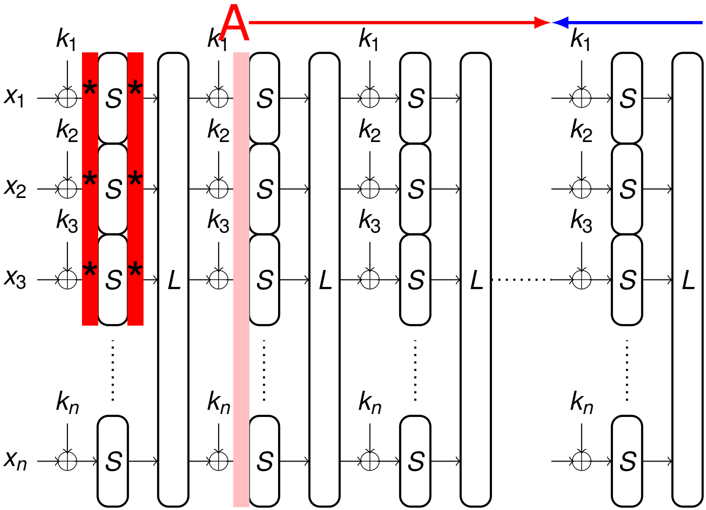

Taking the mode into account

Pyjamask-96

- Distinguisher integral over 10 + 1 rounds
- 3 rounds in the backward direction (monomial count)

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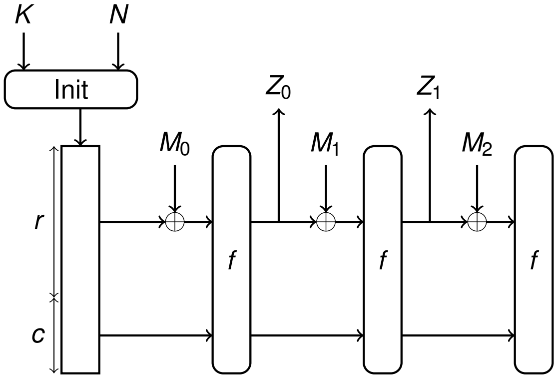
Considering the data complexity...

On duplex or stream ciphers



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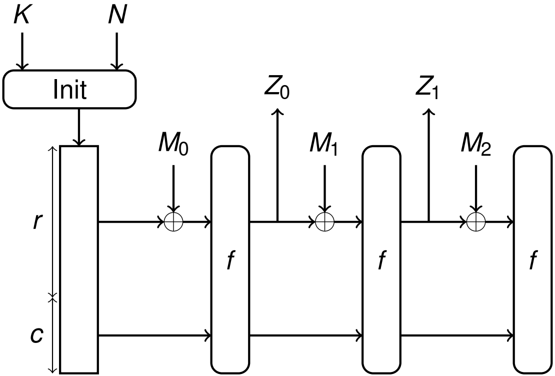

What can you do ?

Representation

Degree

Division property

Attack strategies

Random directions

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- Given a polynomial, find a (non) linear transofrmation that would become an affine space after application ?
- Provide a way to state "every $c_u(k)$ is complicated enough" ?
- Criteria that would be equivalent under the representation of the transformation ?

TAKE AWAY

I'm sure there is a monomial missing somewhere !