Attacks against Filter Generators Exploiting Monomial Mappings

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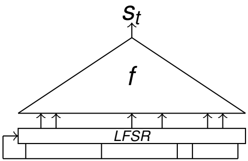
Filtered LFSR

P : the (primitive) characteristic polynomial of the LFSR.
f : nonlinear filtering function.



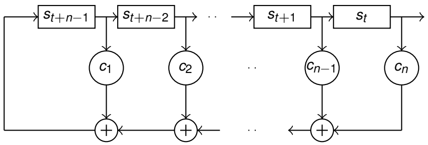
Algebraic Normal Form

$$f(x_1, x_2, \cdots, x_n) = \sum_{u \in \mathbb{F}_2^n} a_u \prod_{i=1}^{n} x_i^{u_i}$$

$$= a_0 + a_1 x_1 + a_2 x_2 + \cdots + a_3 x_1 x_2 + \cdots + a_{2^n-1} x_1 \cdots x_n$$

LFSR over a Finite Field

- α : root of the primitive characteristic polynomial in $\mathbb{F}_{2^n}$
- Identify the n-bit words with elements of $\mathbb{F}_{2^n}$ with the dual basis of $\{1, \alpha, \alpha^2, \cdots, \alpha^{n-1}\}$



Proposition

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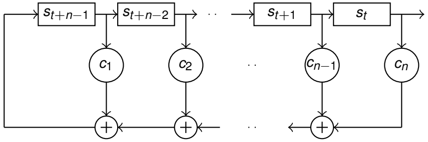

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For all t, $X_t = X_0\alpha^t$

Boolean functions

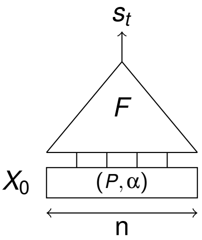
Proposition (Univariate representation)

$$F(X) = \sum_{i=0}^{2^n-1} A_i X^i$$

with $A_i \in \mathbb{F}_{2^n}$ given by the discrete Fourier Transform of F

For all t, $s_t = F(X_0\alpha^t)$

Monomial equivalence [Rønjom - Cid 2010]



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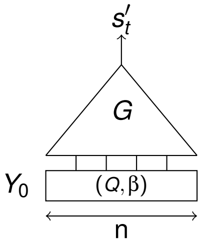

$$\beta = \alpha^k \text{ with } \gcd(k, 2^n - 1) = 1$$

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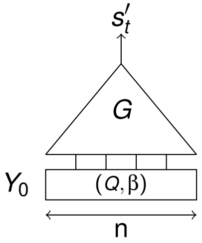

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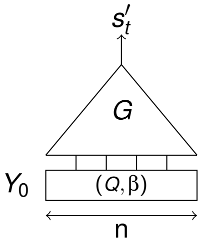

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If $G(x) = F(x^r)$

with $rk \equiv 1 \mod (2^n - 1)$

Then $s'_t = F(Y_0^r \alpha^t)$

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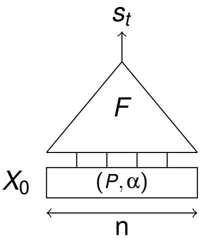

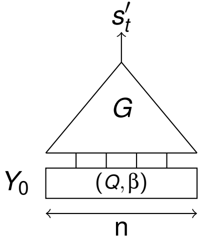

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If $G(x) = F(x^r)$

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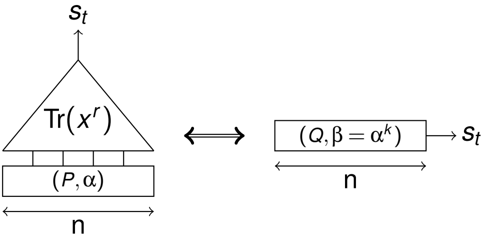
Then $s'_t = F(Y_0^r\alpha^t)$

For all t, $s'_t = s_t$ if $Y_0 = X_0^k$

Example

$F(x) = \mathrm{Tr}(x^r)$, with $\gcd(r, 2^n - 1) = 1$:
Let k be such that $rk \equiv 1 \mod (2^n - 1)$.



$\Longrightarrow$ The initial generator is equivalent to a plain LFSR of the same size.

Consequence

The security level of a filtered LFSR is the minimal security level for a generator of its equivalence class.

- Algebraic attacks
- Correlation attacks

Algebraic attacks

Λ : Linear complexity

Proposition (Massey-Serconek 94)

Let an LFSR of size n filtered by a Boolean function F :

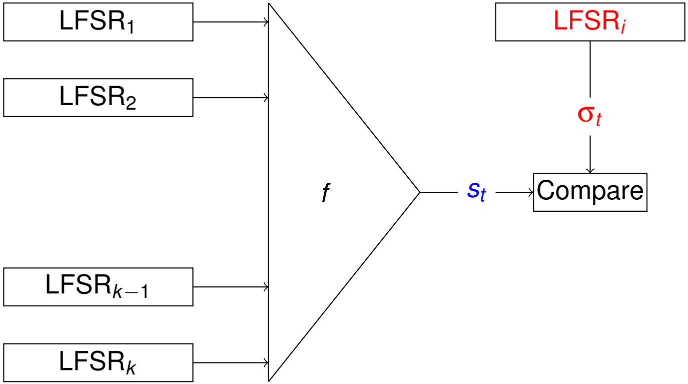
$$F(X) = \sum_{i=0}^{2^n-1} A_i X^i$$

Then

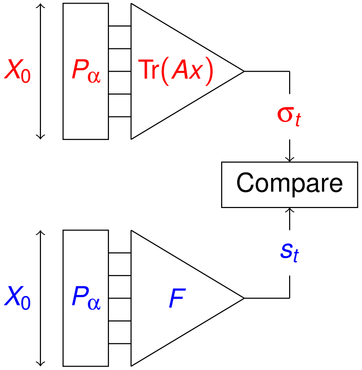
$$\Lambda = \#\{0 \leq i \leq 2^n - 2 : A_i \neq 0\}$$

The monomial equivalence does not affect the complexity of algebraic attacks [Gong et al. 11]

Correlation attack [Siegenthaler 85]

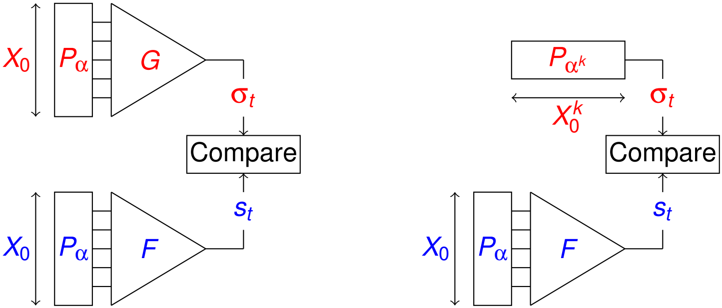


Fast correlation attack [Meier - Staffelbach 88]



Generalized fast correlation attacks

$$G(x) = \mathrm{Tr}(Ax^k)$$



Generalized non-linearity [Gong & Youssef 01]

Relevant security criterion :

Generalized non-linearity

$$\mathrm{GNL}(f) = d(f, \{\mathrm{Tr}(\lambda x^k, \lambda \in \mathbb{F}_{2^n}, \gcd(k, 2^n - 1) = 1\})$$

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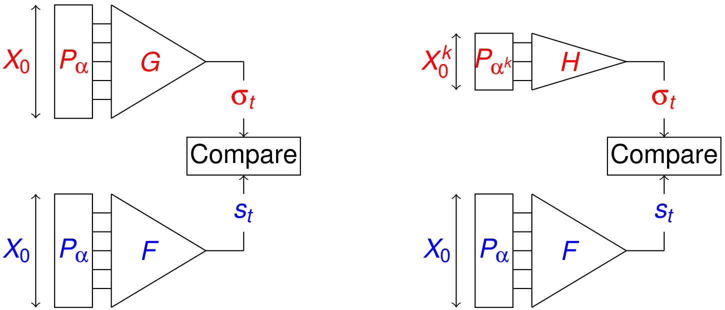
And if k is not coprime to $2^n - 1$?

A more efficient correlation attack

When $\gcd(k, 2^n - 1) > 1$ and F correlated to $G(X) = H(X^k)$.



- Number of states of the small generator : $\tau_k = \mathrm{ord}(\alpha^k)$.
- Exhaustive search on X_0^k : Time $= \frac{\tau_k \log(\tau_k)}{\varepsilon^2}$

Recovering the remaining bits of the initial state

Property

We get $\log_2(\tau_k)$ bits of information on X_0 where $\tau_k = \mathrm{ord}(\alpha^k)$:

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If we perform two distinct correlation attacks with k_1 et k_2, then we get $\log_2(\mathrm{lcm}(\tau_{k_1}, \tau_{k_2}))$ bits of information.

First improvement

The complexity

$$\text{Time} = \frac{\tau_k \log(\tau_k)}{\varepsilon^2}$$

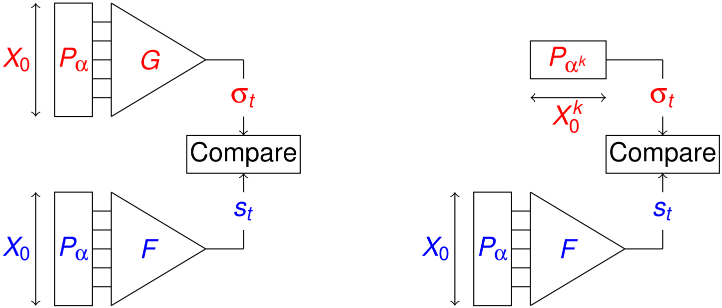
can be reduced to

$$\text{Time} = \tau_k \log(\tau_k) + \frac{2 \log(\tau_k)}{\varepsilon^2} .$$

with a fast Fourier transform [Canteaut - Naya-Plasencia 2012]

Second improvement

$G(X) = H(X^k)$ when H is linear :



- Size of the small LFSR : $L(k) = \text{ord}(2) \mod \tau_k$.
- If $L(k) < n$ and H is linear $\longrightarrow$ fast correlation attack.

Conclusion and open questions

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- Generalized criterion for f besides the generalized non-linearity.
- The attack does not apply when $(2^n - 1)$ is prime.

Open questions

- Find good filtering Boolean functions ?
- Compute efficiently a good approximation of the filtering function ?

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Thank You for your attention !